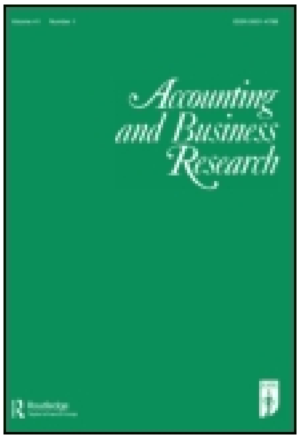

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Non-linear equity valuation

Ali Ataullah [a] , Huw Rhys [b] & Mark Tippett [c]

[a] Senior lecturer in the Business School, Loughborough University

[b] Professor in the Business School, Loughborough University, Leicestershire, LE11 3TU E-mail:

[c] Senior lecturer in the School of Business and Economics, University of Exeter

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Non-linear equity valuation

Ali Ataullah, Huw Rhys and Mark Tippett*

Abstract—We incorporate a real option component into the Ohlson (1995) equity valuation model and then use this augmented model to make assessments about the form and nature of the systematic biases that are likely to arise when empirical work is based on linear models of the relationship between the market value of equity and its determining variables. We also demonstrate how one can expand equity valuation models in terms of an infinite series of 'orthogonal' polynomials and thereby determine the relative contribution which the linear and non-linear components of the relationship between equity value and its determining variables make to overall equity value. This procedure shows that non-linearities in equity valuation can be large and significant, particularly for firms with low earnings-to-book ratios or where the undeflated book value of equity is comparatively small. Moreover, it is highly unlikely the simple linear models that characterise this area of accounting research can form the basis of meaningful statistical tests of the relationship between equity value and its determining variables.

Keywords: branching process; equity valuation, orthogonal polynomials; real option value; recursion value; scaling

1. Introduction

Theoretical developments in capital investment analysis show that real options can play an important role in the capital budgeting process. If a firm has the option of abandoning a poorly performing capital project it can increase the capital project's value well beyond the traditional benchmark given by the present value of its expected future cash flows; likewise, the growth option associated with an unexploited capital project can also be significant when compared with the expected present value of its future cash flows. Moreover, it is now generally accepted that these option values mean that evaluating capital projects exclusively in terms of the present values of their future cash flows can lead to seriously flawed investment decisions – highly profitable capital projects can be overlooked and poorly performing capital projects wrongly implemented. Given this, it is somewhat surprising that both empirical and analytical work on the relationship between equity value and its determining variables continues to be based on models that establish the value of a firm's equity exclusively in terms of the present value of its future operating cash flows and which, therefore, ignore the real option effects associated with the firm's ability to modify or even abandon its existing investment opportunity set. The Ohlson (1995) model, for example, from which much of the empirical work in the area is motivated (Barth and Clinch, 2005: 1) implies that there is a purely linear relationship between the market value of equity and its determining variables. As such, it is based on the implicit assumption that real options are of little significance in the equity valuation process. Given this, it is all but inevitable that when real options do impact on equity values the Ohlson (1995) model will return a systematically biased picture of the relationship between the market value of equity and its determining variables. Fortunately, Ashton et al. (2003) have generalised the Ohlson (1995) model so that it takes account of the real options generally available to firms. Our task here is to use this more general model to determine the likely form and magnitude of the biases that arise under linear equity valuation models like the one formulated by Ohlson (1995).

In the next section we briefly summarise the principal features of the Ashton et al. (2003) equity valuation model and, in particular, some important scale invariance principles on which it is based. Recall here that empirical work in the area is invariably based on market and/or accounting (book) variables that have been normalised or deflated in order to facilitate comparisons between firms of different size. Given this, it is important that one appreciates how these deflation procedures might alter or even distort the underlying levels relationship which exists between the mar-

Ali Ataullah is a senior lecturer and Mark Tippett is a professor in the Business School at Loughborough University. Huw Rhys is a senior lecturer in the School of Business and Economics at the University of Exeter. The authors acknowledge the criticisms and suggestions of the referees whose insightful and probing questions effected a substantial improvement in the manuscript. The paper also benefited from criticisms and suggestions provided by seminar participants in the School of Management at the University of Bradford, the Department of Accounting in the London School of Economics and the 2008 Conference of the British Accounting Association. Such errors as remain, however, are the sole responsibility of the authors. Correspondence should be addressed to Mark Tippett at the Business School, Loughborough University, Leicestershire, LE11 3TU. E-mail: M.Tippett@lboro.ac.uk.

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ket value of equity and its determining variables. In section 3 we introduce a hitherto unused orthogonal polynomial fitting procedure for identifying the relative contribution which the linear and non-linear components of the relationship between equity value and its determining variables make to overall equity value. The evidence from this procedure is that non-linearities in equity valuation can be large and significant, particularly for firms with low earnings-to-book ratios or where the undeflated book value of equity is comparatively small. In Section 4 we examine the specific nature of the statistical biases which arise when the simple linear models that characterise this area of accounting research are used to model the complex non-linear relationships that actually exist. Our conclusion is that it is highly unlikely these simple linear models can form the basis of meaningful statistical tests of the relationship between equity value and its determining variables. Section 5 contains our summary and conclusions.

2. Real options and equity value

The Ashton et al. (2003) model is based on the assumption that the market value of a firm's equity has two components. The first is called the recursion value of equity and is the present value of the future cash flows the firm expects to earn given that its existing investment opportunity set is maintained indefinitely into the future. The Ohlson (1995) equity valuation model is exclusively based on this component of equity value. There is, however, a second component of equity value which the Ohlson (1995) model ignores; namely, the real option (or adaptation) value of equity. This is the option value that arises from a firm's ability to change its existing investment opportunity set by (for example) fundamentally changing the nature of its operating activities (Burgstahler and Dichev, 1997 :188). Ashton et al. (2003) employ this distinction to develop a quasi-supply side generalisation of the Ohlson (1995) model under which the recursion value of equity, $\eta(t)$, evolves in terms of a continuous time branching process (Feller, 1951: 235–237; Cox and Ross, 1976: 149):

$$d\eta(t) = i\eta(t)dt + \sqrt{\eta(t)}dq(t) \quad (1)$$

where i is the cost of equity capital and dq(t) is a Wiener process with a variance parameter of ζ. It then follows that in expectations the proportionate rate of growth in the recursion value of equity will be equal to the cost of equity, or $E_t[d\eta(t)] = i\eta(t)dt$ where $E_t(\cdot)$ is the expectations operator taken at time t. Furthermore, the variance of instantaneous increments in the recursion value of equity will be $Var_t[d\eta(t)] = \zeta\eta(t)dt$. Moreover, since the literature normally describes the firm's investment opportunity set in terms of a first order autoregressive system of information variables it also follows that the recursion value of equity, $\eta(t)$, will be a linear sum of more primitive variables. A good example of this is provided by the Ohlson (1995: 667–669) linear information dynamics which implies that the recursion value of equity evolves in terms of the book value of equity, b(t), book or accounting earnings, x(t), and an information variable, $\nu(t)$, in accordance with the following formula:

$$\eta(t) = c_1 b(t) + c_2 x(t) + c_3 \nu(t) \quad (2)$$

where c_1, c_2 and c_3 are the relevant valuation coefficients.[1]

Ashton et al. (2003) employ the quasi-supply side model summarised above in conjunction with standard no arbitrage conditions and thereby show that the market value of the firm's equity, $P(\eta,B,\theta)$, will have to be:[2]

$$P(\eta,B,\theta) = \eta + \frac{B}{2}\int_{-1}^{1} \exp\left(\frac{-2\theta\eta}{1+z}\right)dz \quad (3)$$

Here B > 0 denotes the value of the firm's adaptation options when the recursion value of equity

[1] There is nothing particularly unique about the Ohlson (1995) linear information dynamics, despite its widespread use in market-based accounting research (Barth and Clinch, 2005: 1). If, for example, one lets earnings and the information variables evolve in terms of a second order system of linear difference equations it is not hard to show that the recursion value of equity will hinge on both the levels of the earnings and information variables as well as the momentum (or first differences) in both these variables. Moreover, if earnings and the information variable evolve in terms of a third order system of linear difference equations, the recursion value of equity will hinge on the levels, momentum and acceleration (or second differences) in both these variables. One could generalise these results to even higher order systems of difference equations. It suffices here to note that these higher order systems provide an analytical basis for the growing volume of empirical evidence which appears to show that the momentum of variables comprising the firm's investment opportunity set can have a significant impact on the value of equity (Chordia and Shivakumar, 2006).

[2] The Ashton et al. (2003) model assumes the firm does not pay dividends and practises 'clean surplus' accounting. However, Ashton et al. (2004) have generalised the Ashton et al. (2003) model so as to relax both these assumptions. They show that the relationship between equity value and recursion value for a dividend paying/dirty surplus firm has exactly the same convex structure and properties as the relationship between equity value and recursion value for a non-dividend paying/clean surplus firm. However, the mathematics of a dividend paying/dirty surplus firm is much more complicated – generally involving numerical procedures as compared to the closed form solutions available under the Ashton et al. (2003) model. Hence, given the pedagogical disadvantages associated with these numerical procedures and the solutions they entail and the equivalent convex structure and properties of the two models, we develop our subsequent analysis in terms of the non-dividend paying/clean surplus Ashton et al. (2003) firm without any loss in generality.

falls away to nothing and

$$\theta = \frac{2i}{Var_t[dq(t)]} = \frac{2i}{\zeta}$$

is a risk parameter that captures the relative stability with which the recursion value of equity evolves over time.[3] Note that η, the first term on the right-hand side of this equation, is the recursion value of equity on which the Ohlson (1995) equity valuation model is exclusively based. We have previously noted, however, that firms invariably have the option of changing their investment opportunity sets and this gives rise to the real option component of equity value captured by the integral term in the above valuation formula. Here it is important to note that as the variability (ζ) of the recursion value increases relative to the cost of equity (i), the term $\exp(\frac{-2\theta\eta}{1+z})$ grows in magnitude and the real option value of equity becomes larger as a consequence. Similarly, the real option value of equity falls as the variability of the recursion value declines relative to the cost of equity. In other words, when the rate of growth in recursion value clusters closely around the cost of equity it is unlikely the catastrophic events which will induce the firm to exercise its real options will arise. In these circumstances, the small probability of these options ever being exercised will mean that the real option value of equity will also have to be comparatively small.

Here we need to note, however, that empirical work in the area is invariably based on market and/or accounting (book) variables that have been normalised or deflated in order to facilitate comparisons between firms of different size. Given this, suppose one defines the normalised recursion value, $h(t) = \frac{\eta(t)}{w}$, where w is the normalising factor. It then follows that increments in the normalised recursion value will evolve in terms of the process:

$$dh(t) = ih(t)dt + \sqrt{h(t)}dv(t) \tag{4}$$

where

$$dv(t) = \frac{1}{\sqrt{w}}.dq(t)$$

is a Wiener process with variance parameter $\frac{\zeta}{w}$. Note that increments in the normalised recursion value, h(t), will have a mean and variance of $E_t[dh(t)] = ih(t)dt$ and $Var_t[dh(t)] = h(t)Var_t[dv(t)] = h(t).\frac{\zeta}{w}dt$, respectively. This in turn implies that the risk parameter for the normalised recursion value will be

$$\frac{2i}{Var_t[dv(t)]} = \frac{2iw}{\zeta} = \theta w.$$

Furthermore, if one works in terms of this normalised variable, h(t), rather than in levels, η(t), it follows that the value of equity will have to be:

$$P(h, \frac{B}{w}, \theta w) = h + \frac{B}{2w}\int_{-1}^{1} \exp(\frac{-2\theta wh}{1+z})dz \tag{5}$$

Here it is important to note that η(t) = wh(t) and so, the normalised value of equity will satisfy the following important property:

$$wP(h, \frac{B}{w}, \theta w) = \eta + \frac{B}{2}\int_{-1}^{1} \exp(\frac{-2\theta\eta}{1+z})dz = P(\eta, B, \theta) \tag{6}$$

Formally, this result means that the Ashton et al. (2003) equity valuation model is scale-invariant under all dilations, w (Borgnat et al., 2002: 181).[4]

Now consider a firm for which all variables have been deflated by the book value of equity, w = B, as at some *fixed* date or that $h(t) = \frac{\eta(t)}{B}$ in the scaled version of the Ashton et al. (2003) model given earlier. Moreover, assume, for purposes of illustration, that the firm's adaptation options involve selling off its existing investment opportunities at their book values as recorded on the balance sheet on this fixed date and using the proceeds to move into alternative lines of business.[5] These consider-

[3] The Ashton et al. (2003) model follows previous work (Burgstahler and Dichev, 1997: 190–195) in assuming that the adaptation value of equity, B, is a given known constant. There are, however, a variety of ways in which one could allow B to vary stochastically in the Ashton et al. (2003) valuation formula given here. The most parsimonious of these lets B evolve as a random walk – although even here one could have an error structure that allows for the variance of increments in adaptation value to hinge on the current adaptation value itself, thereby preventing adaptation value from becoming negative. Standard no-arbitrage conditions will then lead to exactly the same Hamilton-Jacobi-Bellman equation from which the Ashton et al. (2003) valuation formula given above is derived.

[4] If, for all real x, λ and some Δ, a function *f*(·), satisfies the property $\lambda^{\Delta}f(x) = f(\lambda x)$ then it is said to be scale invariant for all dilations, λ, with a scaling dimension of Δ (Borgnat, et al., 2002: 181). See Alexander and Nogueira (2006: 4–6) for examples of option models that violate the scale-invariance property given here.

[5] This assumption is widely used and accepted in the empirical work of the area. Burgstahler and Dichev (1997: 195), for example, define the book value of equity at the beginning of the interval covered by the profit and loss statement 'as the measure of adaptation value for the year'. Barth et al. (1998: 1–2), Collins et al. (1999: 32), Ashton et al. (2003: 427) and Cotter and Donnelly (2006: 11) amongst many others also invoke this assumption. All these authors base their argument for using opening book value as the adaptation value for the year on the fact that the impairment testing procedures summarised in accounting standards like FRS 10: *Goodwill and Intangible Assets*, FRS 11: *Impairment of Fixed Assets and Goodwill*, FRS 15: *Tangible Fixed Assets* and the Fourth Schedule of the Companies Act (or their North American equivalents) require assets appearing on a firm's balance sheet to be carried at no more than their recoverable amounts. This in turn means it is highly likely the book value of equity at some given fixed point in time will constitute a reasonable approximation for the adaptation value of equity over short intervals surrounding that fixed point in time. See Burgstahler and Dichev (1997: 187–194) for an elaboration of these arguments.

ations will mean the market value of equity (per unit of book value) evolves in terms of the following formula:

$$\frac{P(\eta,B,\theta,)}{B} = h + \frac{1}{2}\int_{-1}^{1} \exp\left(\frac{-2\theta Bh}{1+z}\right)dz = P(h,1,\theta B) \quad (7)$$

where $\frac{P(\eta,B,\theta)}{B}$ is the ratio of the market value of equity to its book value on the given fixed date. This result implies that there will be a convex relationship between the market-to-book ratio of equity, P(h,1,θB), and the recursion value of equity divided by the book value of equity, h.[6] Suppose that one assumes the existence of a linear relationship between the market value of equity and its determining variables, as is the case with the Ohlson (1995) model. These linear models are based on the implicit assumption that the real options generally available to firms have no role to play in the equity valuation process. However, if one uses a linear model to approximate the relationship between equity value and its determining variables when, in fact, real options do impact on equity values then it is all but inevitable there will be systematic differences between the actual market values and those predicted by the linear model. One can illustrate the point being made here by approximating the market-to-book ratio given in the scaled version of the Ashton et al. (2003) model by a linear function over its entire domain. Our approximating procedures make use of an inner product (Hilbert) space using the Laguerre polynomials as a basis and are summarised in further detail in the Appendix. These procedures show that the best linear approximation to the equity valuation function [P(h,1,θB)] over the semi-infinite real line will be:

$$P(h,1,\theta B) = h + \frac{1}{2}\int_{-1}^{1} \exp\left(\frac{-2\theta Bh}{1+z}\right)dz \approx \frac{1}{1+\theta B} + \left[1 + \frac{\theta B}{1+\theta B} + \theta B\log\left(\frac{\theta B}{1+\theta B}\right)\right].h \quad (8)$$

As a particular example, consider a firm whose normalised risk parameter is θB = 2 in which case substitution shows that the best linear approximation to the equity valuation function will be:[7]

$$P(h,1,2) = h + \frac{1}{2}\int_{-1}^{1} \exp\left(\frac{-4h}{1+z}\right)dz \approx 0.3333 + 0.8557h \quad (9)$$

Figure 1 contains a diagrammatic summary of these results. The upward sloping line emanating from the origin at a 45 degree angle is the normalised recursion value of equity, $h = \frac{\eta}{B}$. The downward sloping curve which asymptotes towards the recursion value axis is the normalised real option value of equity,

$$\frac{1}{2}\int_{-1}^{1} \exp\left(\frac{-4h}{1+z}\right)dz.$$

The sum of the normalised recursion and normalised real option values is the market value of equity per unit of book value,

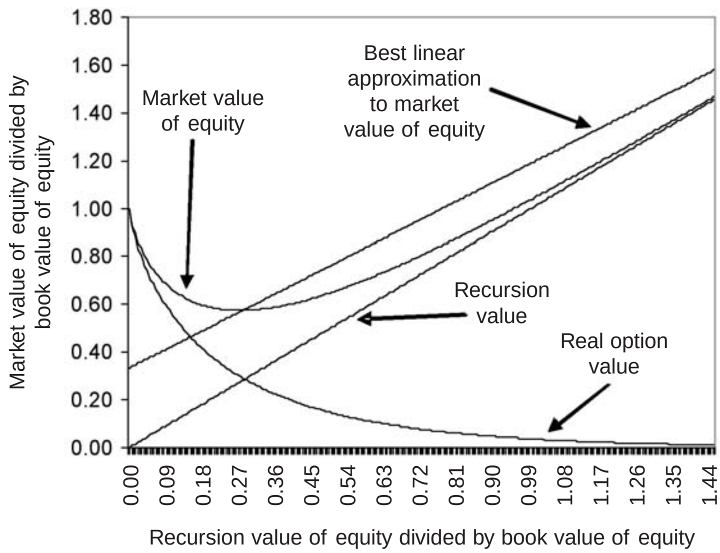
$$P(h,1,2) = h + \frac{1}{2}\int_{-1}^{1} \exp\left(\frac{-4h}{1+z}\right)dz,$$

and is represented by the convex curve which asymptotes towards the 45 degree line representing the normalised recursion value of equity. Here it is important to observe that as the normalised recursion value increases in magnitude, the market value of equity (per unit of book value) at first declines before reaching a minimum and then gradually increases in magnitude. This arises because at small recursion values the decline in real option value will be much larger than the increase in the recursion value itself. This, in turn, means that the best linear approximation, 0.3333 + 0.8557h, to the overall market value of equity will bear a particular relationship to the market value of equity, P(h,1,2). Hence, when the normalised recursion value of equity is low ($h < 0.30$) then the difference between the linear approximation and the market value of equity, (0.3333 + 0.8557h) – P(h,1,2), is negative. Beyond this point (that is, $h > 0.30$) the difference between the linear approximation and the market value of equity becomes positive before falling away again and becoming negative when

[6] The early evidence of Burgstahler and Dichev (1997: 199–205) and Burgstahler (1998: 339) shows that at an aggregate level there is a highly non-linear and generally convex relationship between the ratio of the market value of equity to the book value of equity and the ratio of earnings attributable to equity to the book value of equity for US equity securities over the 20-year period ending in 1994. Likewise, Ashton et al. (2003: 429–430) show that an almost identical convex relationship exists at an aggregate level between the market value of equity and the earnings attributable to equity for UK equity securities over the period from 1987 to 1998. Finally, Di-Gregorio (2006) shows that there is a highly convex relationship (again at an aggregate level) between the market value of equity and earnings for German and Italian firms over the period from 1995 to 2005. Moreover, our own unreported empirical work covering 18 broad based UK industrial classifications over the period from 2001 to 2004 shows that the convex relationship obtained at an aggregate level by previous authors also consistently holds at an industry level. Hence, there is now overwhelming international empirical evidence in support of the hypothesis that there is a highly non-linear and generally convex relationship between earnings and the market value of equity.

[7] Ataullah et al. (2006) summarise empirical evidence which is broadly compatible with this value of the normalised risk parameter, θB.

Figure 1
Plot of recursion value of equity, real option value of equity, overall market value of equity and linear approximation to overall value of equity for a branching process with risk parameter θB = 2



The upward sloping line emanating from the origin at a 45 degree angle is the normalised recursion value of equity, h. The downward sloping curve which asymptotes towards the recursion value axis is the normalised real option value of equity,

$$\frac{1}{2}\int_{-1}^{1}\exp\left(\frac{-4h}{1+z}\right)dz.$$

The sum of the normalised recursion and real option values is the total market value of equity divided by the book value of equity,

$$P(h,1,2) = h + \frac{1}{2}\int_{-1}^{1}\exp\left(\frac{-4h}{1+z}\right)dz,$$

and is represented by the convex curve which asymptotes towards the 45 degree line representing the normalised recursion value of equity. The line emanating from the point 0.3333 on the market value axis is the best linear approximation, $P(h,1,2) \approx 0.3333 + 0.8557h$, to the overall market value of equity.

$h > 2.31$ (although this is not shown on the graph). In other words, for individual firms there will be systematic biases in the linear model which is used to approximate the relationship between the market value of equity and its determining variables.[8]

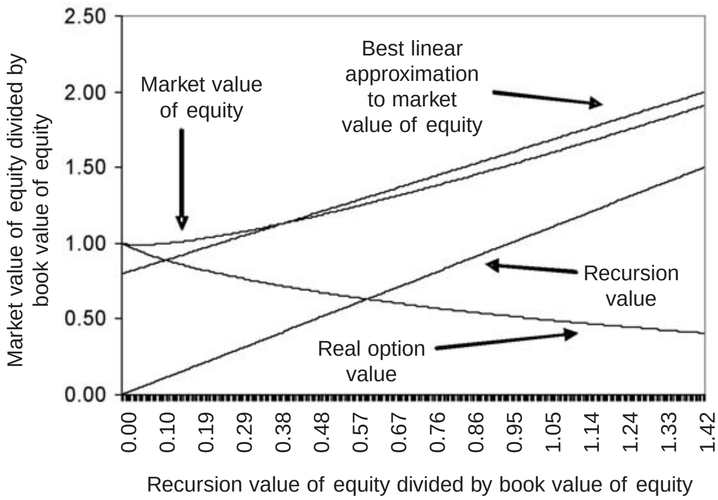
Here we need to emphasise, however, that the form and nature of the systematic biases which arise from approximating the relationship between equity value and its determining variables in terms of a linear model will very much depend on the magnitude of the normalised risk parameter, θB. The smaller this parameter (and by implication the larger real option values) the more likely it is that a linear model will provide a good approximation to the relationship between the market value of equity and its determining variables. This is illustrated by Figure 2 which shows for a firm whose normalised risk parameter is θB = 0.25, that the best linear approximation to the equity valuation function will be:

$$P(h,1,0.25) = h + \frac{1}{2}\int_{-1}^{1}\exp\left(\frac{-0.5h}{1+z}\right)dz \approx 0.8000 + 0.7976h \tag{10}$$

Note that with a relatively small normalised risk

[8] It is somewhat surprising that linear valuation models continue to dominate analytical and empirical work in this area given that Black and Scholes (1973: 649–652) and Cox and Ross (1976: 163–165) demonstrated over 30 years ago that it is highly unlikely such models can provide an adequate description of the way equity prices evolve in practice.

Figure 2
Plot of recursion value of equity, real option value of equity, overall market value of equity and linear approximation to overall value of equity for a branching process with risk parameter θB = 0.25



The upward sloping line emanating from the origin at a 45 degree angle is the normalised recursion value of equity, h. The downward sloping curve which asymptotes towards the recursion value axis is the normalised real option value of equity,

$$\frac{1}{2}\int_{-1}^{1}\exp\left(\frac{-0.5h}{1+z}\right)dz.$$

The sum of the normalised recursion and real option values is the total market value of equity divided by the book value of equity,

$$P(h,1,0.25) = h + \frac{1}{2}\int_{-1}^{1}\exp\left(\frac{-0.5h}{1+z}\right)dz,$$

and is represented by the convex curve which asymptotes towards the 45 degree line representing the normalised recursion value of equity. The line emanating from the point 0.8000 on the market value axis is the best linear approximation, $P(h,1,0.25) \approx 0.8000 + 0.7976h$, to the overall market value of equity.

parameter like this it is only at very small and very large ratios of the recursion to book value of equity that linear approximations will provide a poor reflection of the relationship between the market value of equity and its determining variables.

One can further illustrate the importance of the systematic biases demonstrated in these examples by thinking of the valuation equation $P(h,1,\theta B)$ as a representative firm in a large cross-sectional sample of similarly prepared firms. By 'similarly prepared' is meant that all firms are characterised by a common investment opportunity set and are, therefore, described by a common equity valuation function; namely, $P(h,1,\theta B)$.[9] It then follows that if equity values include a real option component, cross sectional linear regression models of the relationship between equity prices and recursion values will follow a pattern similar to that obtained for the above examples. That is, one would expect to find firms with low ratios of recursion value to the book value of equity returning negative residuals from a linear regression model. Likewise, firms with intermediate ratios of recursion value to the book value of equity will return positive residuals from the linear regression model. Finally, when the ratio of recursion value to the book value of equity is large, one would expect to see negative residuals again emerging from the linear regression model.[10]

[9] This approach underscores much of the empirical work conducted in the area. Dechow et al. (1999) and Collins, et al. (1999), Morel (2003) and Gregory et al. (2005) provide some recent examples.

3. The relative importance of non-linearities in equity valuation

Our previous analysis establishes the inevitability of systematic biases in models that presume a purely linear relationship between the market value of equity and its determining variables. We now investigate whether it is possible to isolate the contribution which the linear and non-linear components of the relationship between equity value and its determining variables make to overall equity value and, in particular, whether it might be possible to characterise equity value in terms of a low order polynomial expansion of its determining variables. We begin by noting that the inner product (Hilbert) space framework employed earlier can be used to express equity value as an infinite series of orthogonal (that is, uncorrelated) polynomial terms of the determining variables and this in turn, allows one to ascertain the relative contribution which each polynomial term makes to the overall variation in equity value. One can illustrate the point being made here by noting that the inner product (Hilbert) space framework implies that the Ashton et al. (2003) equity valuation formula, P(h,1,θB), can be expressed in terms of an infinite series expansion of Laguerre polynomials, namely:[11]

$$P(h,1,\theta B) = \sum_{n=0}^{\infty} \alpha_n L_n(h) \tag{12}$$

where $L_0(h) = 1, L_1(h) = 1 - h$ and when $n \geq 2$, $nL_n(h) = (2n - 1 - h)L_{n-1}(h) - (n - 1)L_{n-2}(h)$ are the Laguerre polynomials and

$$\alpha_0 = \theta B\log\left(\frac{\theta B}{1 + \theta B}\right) + 2,\ \alpha_1 = -1 - \frac{\theta B}{1 + \theta B} - \theta B\log\left(\frac{\theta B}{1 + \theta B}\right)$$

and when $n \geq 2$,

$$\alpha_n = \frac{\theta B(1 + \theta B)^n - (\theta B)^n(n + \theta B)}{n(n - 1)(1 + \theta B)^n}$$

are the coefficients associated with each of the Laguerre polynomials in the series expansion.

Now suppose one approximates the equity valuation function as a linear sum of the first (m + 1) Laguerre polynomials, or:

$$P(h,1,\theta B) \approx \sum_{n=0}^{m} \alpha_n L_n(h) \tag{13}$$

It can then be shown that:

$$\alpha_0^2 . [\sum_{n=0}^{\infty} \alpha_n^2)]^{-1}$$

gives the proportion of the squared variation in the equity valuation function, P(h,1,θB), which is accounted for by the Laguerre polynomial $L_0(h) = 1$ (Apostol, 1967: 566). Likewise,

$$\alpha_1^2 . [\sum_{n=0}^{\infty} \alpha_n^2)]^{-1}$$

gives the proportion of the squared variation in the equity valuation function which is accounted for by the Laguerre polynomial $L_1(h) = 1 - h$. Similarly,

$$\alpha_2^2 . [\sum_{n=0}^{\infty} \alpha_n^2)]^{-1}$$

gives the proportion of the squared variation which is accounted for by the Laguerre polynomial $L_2(h) = \frac{1}{2}(h^2 - 4h + 2)$. Continuing this procedure shows that:

$$R_m^2 = \sum_{n=0}^{m} \alpha_n^2 . [\sum_{n=0}^{\infty} \alpha_n^2)]^{-1} \tag{14}$$

gives the proportion of the squared variation in the equity valuation function which is accounted for by the first (m + 1) Laguerre polynomials. We now show that one can use these results to determine the relative contribution which the linear and non-linear components of the equity valuation function make to overall equity value.[12]

Table 1 summarises the relative contribution which the Laguerre polynomials of order m = 0 to m = 100 make to the overall squared variation of the Ashton et al. (2003) equity valuation function,

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[10] If one has a large sample of firms then it is not hard to show that under the Ohslon (1995) model a graph of the market value of equity against earnings will, except for a scaling factor, be equivalent to the graph of the market value of equity against its recursion value. Recall here that the Ohlson (1995) model predicts a linear relationship between equity value and earnings in contrast to the highly non-linear and generally convex relationship that exists between these variables in the empirical work summarised by Burgstahler and Dichev (1997), Burgstahler (1998), Ashton et al. (2003) and, Di-Gregorio (2006) amongst others. The convex relationship obtained in this empirical work provides strong indirect evidence that real options have a significant role to play in the determination of equity values.

[11] The proofs for this and subsequent results appear in the Appendix.

[12] The procedure articulated here is equivalent to the spectral decomposition of a variance-covariance matrix using the method of Principal Components (Rao, 1964). The squared coefficients, α_n^2, are equivalent to the eigenvalues of the variance-covariance matrix. In the same way as the ratio of a given eigenvalue to the sum of all eigenvalues gives the proportion of the variance accounted for by the eigenvector (principal component) corresponding to the given eigenvalue, the ratio of the squared coefficient to the sum of all squared coefficients gives the proportion of the squared variation in the equity valuation function accounted for by the particular Laguerre polynomial.

Table 1
Proportion of squared variation in equity valuation function associated with Laguerre polynomials

	θB = 0.25			θB = 0.50			θB = 0.75			θB = 1.00		
m	α_m	$\alpha_m^2 \cdot [\sum_{n=0}^{\infty} \alpha_n^2)]^{-1}$	R_m^2	α_m	$\alpha_m^2 \cdot [\sum_{n=0}^{\infty} \alpha_n^2)]^{-1}$	R_m^2	α_m	$\alpha_m^2 \cdot [\sum_{n=0}^{\infty} \alpha_n^2)]^{-1}$	R_m^2	α_m	$\alpha_m^2 \cdot [\sum_{n=0}^{\infty} \alpha_n^2)]^{-1}$	R_m^2
0	1.5976	0.798338	0.798338	1.4507	0.768643	0.768643	1.3645	0.739860	0.739860	1.3069	0.714979	0.714979
1	–0.7976	0.198996	0.997334	–0.7840	0.224510	0.993153	–0.7931	0.249942	0.989802	–0.8069	0.272539	0.987518
2	0.0800	0.002002	0.999336	0.1111	0.004509	0.997662	0.1224	0.005958	0.995760	0.1250	0.006541	0.994059
3	0.0373	0.000436	0.999772	0.0617	0.001392	0.999054	0.0758	0.002283	0.998044	0.0833	0.002907	0.996966
4	0.0203	0.000128	0.999900	0.0370	0.000501	0.999555	0.0491	0.000960	0.999003	0.0573	0.001374	0.998341
5	0.0124	0.000048	0.999948	0.0239	0.000208	0.999763	0.0333	0.000442	0.999445	0.0406	0.000691	0.999031
6	0.0083	0.000022	0.999970	0.0164	0.000098	0.999861	0.0236	0.000221	0.999667	0.0297	0.000369	0.999400
7	0.0060	0.000011	0.999981	0.0118	0.000051	0.999912	0.0174	0.000120	0.999786	0.0223	0.000209	0.999609
8	0.0045	0.000006	0.999987	0.0089	0.000029	0.999941	0.0132	0.000069	0.999856	0.0172	0.000124	0.999733
9	0.0035	0.000004	0.999991	0.0069	0.000018	0.999958	0.0104	0.000043	0.999898	0.0136	0.000078	0.999811
10	0.0028	0.000002	0.999994	0.0056	0.000011	0.999970	0.0083	0.000027	0.999926	0.0110	0.000051	0.999862
15	0.0012	0.000000	0.999998	0.0024	0.000002	0.999991	0.0036	0.000005	0.999978	0.0048	0.000009	0.999959
30	0.0003	0.000000	1.000000	0.0006	0.000000	0.999999	0.0009	0.000000	0.999997	0.0011	0.000001	0.999995
50	0.0001	0.000000	1.000000	0.0002	0.000000	1.000000	0.0003	0.000000	0.999999	0.0004	0.000000	0.999999
75	0.0000	0.000000	1.000000	0.0001	0.000000	1.000000	0.0001	0.000000	1.000000	0.0002	0.000000	1.000000
100	0.0000	0.000000	1.000000	0.0001	0.000000	1.000000	0.0001	0.000000	1.000000	0.0001	0.000000	1.000000

Table 1
Proportion of squared variation in equity valuation function associated with Laguerre polynomials *(continued)*

	θB = 2			θB = 4			θB = 6			θB = 8		
m	α_m	$\alpha_m^2.[\sum_{n=0}^{\infty}\alpha_n^2)]^{-1}$	R_m^2	α_m	$\alpha_m^2.[\sum_{n=0}^{\infty}\alpha_n^2)]^{-1}$	R_m^2	α_m	$\alpha_m^2.[\sum_{n=0}^{\infty}\alpha_n^2)]^{-1}$	R_m^2	α_m	$\alpha_m^2.[\sum_{n=0}^{\infty}\alpha_n^2)]^{-1}$	R_m^2
0	1.1891	0.648684	0.648684	1.1074	0.590307	0.590307	1.0751	0.564577	0.564577	1.0577	0.550209	0.550209
1	–0.8557	0.335968	0.984652	–0.9074	0.396343	0.986650	–0.9322	0.424505	0.989082	–0.9466	0.440686	0.990896
2	0.1111	0.005664	0.990316	0.0800	0.003081	0.989731	0.0612	0.001831	0.990913	0.0494	0.001199	0.992095
3	0.0864	0.003426	0.993743	0.0693	0.002314	0.992044	0.0554	0.001499	0.992412	0.0457	0.001028	0.993123
4	0.0679	0.002115	0.995858	0.0603	0.001748	0.993793	0.0502	0.001230	0.993642	0.0424	0.000883	0.994006
5	0.0539	0.001333	0.997191	0.0525	0.001329	0.995122	0.0455	0.001013	0.994655	0.0393	0.000759	0.994765
6	0.0433	0.000858	0.998050	0.0460	0.001016	0.996138	0.0414	0.000836	0.995491	0.0365	0.000654	0.995420
7	0.0351	0.000565	0.998614	0.0403	0.000782	0.996920	0.0376	0.000692	0.996183	0.0339	0.000565	0.995984
8	0.0287	0.000379	0.998994	0.0355	0.000606	0.997526	0.0343	0.000575	0.996758	0.0315	0.000488	0.996472
9	0.0238	0.000260	0.999253	0.0313	0.000472	0.997998	0.0313	0.000479	0.997236	0.0293	0.000423	0.996895
10	0.0199	0.000182	0.999435	0.0277	0.000370	0.998369	0.0286	0.000400	0.997636	0.0273	0.000367	0.997261
15	0.0093	0.000040	0.999821	0.0159	0.000121	0.999350	0.0187	0.000170	0.998861	0.0194	0.000185	0.998494
30	0.0023	0.000002	0.999978	0.0045	0.000010	0.999908	0.0065	0.000021	0.999798	0.0079	0.000031	0.999669
50	0.0008	0.000000	0.999996	0.0016	0.000001	0.999982	0.0024	0.000003	0.999959	0.0032	0.000005	0.999928
75	0.0004	0.000000	0.999999	0.0007	0.000000	0.999996	0.0011	0.000001	0.999992	0.0014	0.000001	0.999986
100	0.0002	0.000000	1.000000	0.0004	0.000000	1.000000	0.0006	0.000000	1.000000	0.0008	0.000000	1.000000

The above table gives the mth order Laguerre coefficient, α_m, and the proportion,

$$\alpha_m^2.[\sum_{n=0}^{\infty}\alpha_n^2)]^{-1},$$

of the squared variation in the equity valuation function, P(h,1,θB), which is accounted for by the given Laguerre polynomial. It also gives the accumulated proportion, R^2_m, of the squared variation in the equity valuation function which is accounted for by the first m Laguerre polynomials.

P(h,1,θB), for values of the scaled risk parameter that vary from θB = 0.25 to θB = 8.[13] Note how the Table shows that a linear approximation (R_1^2), based on the coefficients α_0 and α_1, accounts for over 98% of the squared variation of the Ashton et al. (2003) equity valuation function, irrespective of the scaled risk parameter, θB, on which the approximation is based. If, for example, one follows the analysis in Section 2 by letting θB = 2 then Table 1 shows $\alpha_0 = 1.1891$ and $\alpha_1 = -0.8557$ or that consistent with equation (9), the best linear approximation to the equity valuation function is:

$$P(h,1,2) = h + \frac{1}{2}\int_{-1}^{1} \exp\left(\frac{-4h}{1+z}\right)dz$$

$$\approx \alpha_0 + \alpha_1(1 - h) = 0.3333 + 0.8557h$$

Moreover, Table 1 also shows that this simple linear approximation accounts for $R_1^2 = 98.4652\%$ of the squared variation in P(h,1,2). This has the important implication that the non-linear terms in the polynomial expansion account for no more than $1 - R_1^2 = 1 - 0.984652 = 1.5348\%$ of the squared variation in P(h,1,2). A similar conclusion applies for other values of the scaled risk parameter, θB, summarised in Table 1; the non-linear terms in the polynomial expansion make only a minor contribution to the squared variation in the equity valuation function. Given this, one might conclude that a linear approximation of the relationship between equity value and its determining variables will suffice for most practical purposes. Unfortunately, the least squares procedures employed here and also in most of the empirical work of the area, suffer from a ringing artifact known as Gibbs' phenomenon.[14] In the present context Gibbs' phenomenon implies that the Laguerre series expansion will display irregular behaviour in an arbitrarily small interval near the origin (Fay and Kloppers, 2006). This, in turn, means that whilst a linear approximation will provide generally reasonable estimates of the equity valuation function when the recursion value of equity is comparatively large, it will, unfortunately, perform poorly near the origin (that is, when the recursion value of equity is relatively small). It is in this latter circumstance that one will need to include higher order terms from the series expansion if there is to be any prospect of obtaining reasonable approximations to the equity valuation function – something that is borne out by the empirical work summarised in the literature (Burgstahler and Dichev, 1997; Burgstahler, 1998; Ashton et al., 2003 and Di-Gregorio, 2006).

The exact degree to which the Laguerre polynomial series expansion must be carried before one obtains reasonable approximations to the equity valuation function near the origin very much depends on the magnitude of the scaled risk parameter, θB. Larger values of this parameter will generally require the inclusion of higher order polynomial terms. Here one can again follow the analysis in Section 2 in letting θB = 2 in which case Figure 3 plots the equity valuation function, P(h,1,2), together with its fifth (m = 5), tenth (m = 10), fifteenth (m = 15) and twentieth (m = 20) degree Laguerre polynomial series approximations. Note how the fifth degree Laguerre approximation is particularly poor near the origin and that the tenth and fifteenth degree approximations, whilst an improvement, are still not entirely satisfactory. Indeed, it is only when one employs a twentieth degree polynomial expansion that the approximation to the equity valuation function becomes at all reasonable near the origin. The important point here is that even though linear approximations may appear to be more than reasonable over virtually the entire domain of equity values, nonetheless near the origin (where the recursion value of equity is small) they can be especially poor. This means that mis-specification errors are more likely with samples comprised of firms with comparatively small recursion values – for example, those threatened with administration or which are experiencing other forms of financial distress (Barth et al., 1998). In such instances it is doubtful whether linear models of the relationship between the market value of equity and its determining variables can adequately capture the empirical relationships which exist in the area.[15]

4. The biases in simple linear models of equity valuation

Additional insights into the nature of the biases that arise from the simple linear models that pervade this area of accounting research can be obtained by supposing one has a large cross-sectional sample of similarly prepared firms. Let the market value of equity for these j = 1,2,3,___, n firms be $P_j = P_j(h,1,\theta B)$ and its associated recursion value be η_j. It then follows that for each of these firms the relationship between the market value of equity and its recursion value will be:[16]

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[13] We have previously noted that Ataullah et al. (2006) summarise empirical evidence which is broadly compatible with these values of the normalised risk parameter, θB.

[14] The 'least squares' techniques on which much of the empirical work of the area is based can also be formalised in terms of an inner product space with a Euclidean norm and will, as a consequence, also be affected by Gibbs' phenomenon.

[15] Empiricists will often (implicitly) acknowledge the non-linear nature of the valuation relationships that exist in these situations by using dummy variables to allow regression coefficients to vary with so called 'extreme' observations (Barth et al., 1998: 5–9). However, whether this procedure can satisfactorily address the omitted variables problems implicit in their empirical work has yet to be demonstrated.

Figure 3
Plot of polynomial approximations to overall market value of equity for branching process with risk parameter θB = 2

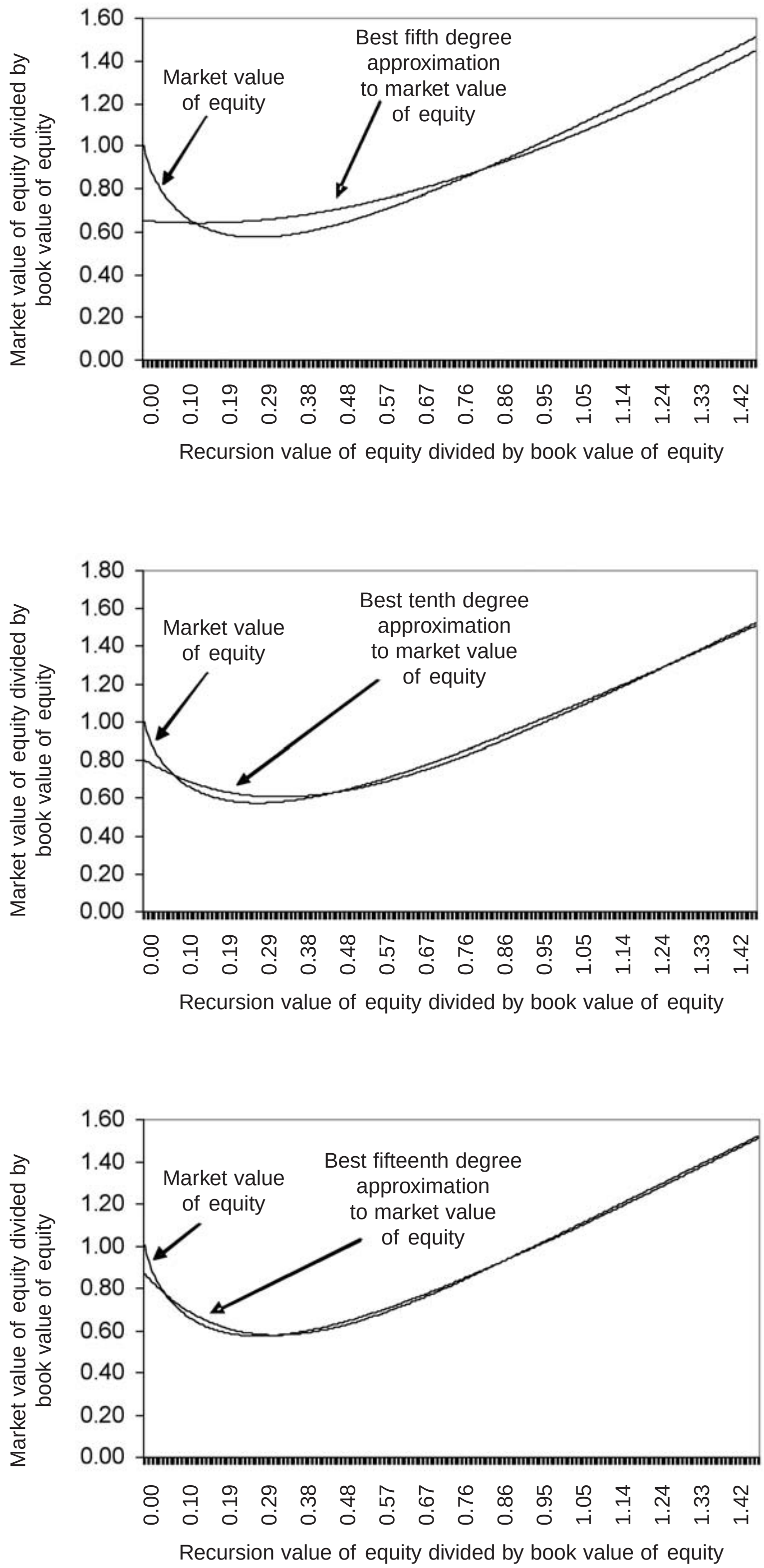
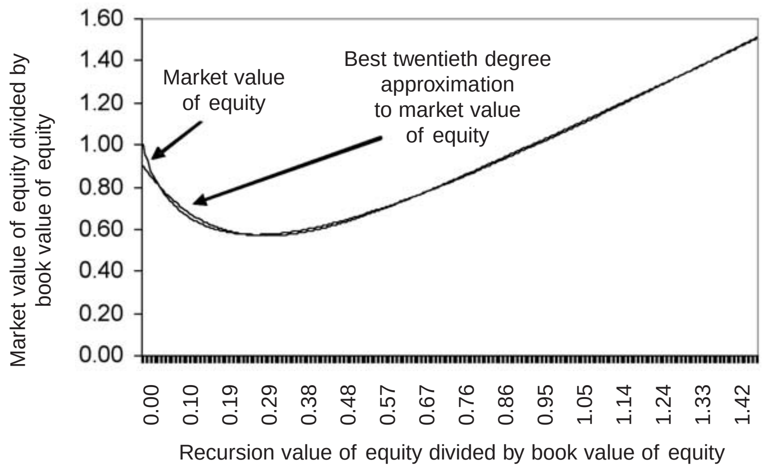


Figure 3
Plot of polynomial approximations to overall market value of equity for branching process with risk parameter θB = 2 *(continued)*



The two curves in the above graphs are firstly, the market value of equity divided by the book value of equity,

$$P(h,1,2) = h + \frac{1}{2}\int_{-1}^{1} \exp\left(\frac{-4h}{1+z}\right)dz$$

The second curve is the Laguerre polynomial approximation

$$P(h,1,2) \approx \sum_{n=0}^{m} \alpha_n L_n(h)$$

for m = 5 (first graph), m = 10 (second graph), m = 15 (third graph) and m = 20 (fourth graph).

$$P_j = \sum_{k=1}^{m} \beta_k \eta_j^k + e_j \qquad (15)$$

where m is the order of the polynomial expansion, β_k is the coefficient associated with the k^{th} polynomial term in η_j and e_j is the error term associated with the approximation. Now, suppose one fits a linear model, $P_j = \beta\eta_j$, to the given data where β is a fixed parameter (Hayn, 1995; Kothari and Zimmerman, 1995). It then follows that the estimate of β will be:

$$\hat{\beta} = \frac{\sum_{j=1}^{n} \eta_j . P_j}{\sum_{j=1}^{n} \eta_j^2} = \beta_1 + \frac{\sum_{j=1}^{n} [\beta_2\eta_j^3 + \beta_3\eta_j^4 + \_\_\_\_\_\_ + \beta_m\eta_j^{m+1} + \eta_j e_j]}{\sum_{j=1}^{n} \eta_j^2} \qquad (16)$$

Hence, if one follows conventional practice in assuming the error term, e_j, has a mean of zero then one will obtain the following estimate of β on average:

$$E(\hat{\beta}) = \beta_1 + \frac{\sum_{j=1}^{n} [\beta_2\eta_j^3 + \beta_3\eta_j^4 + \_\_\_\_\_\_ + \beta_m\eta_j^{m+1}]}{\sum_{j=1}^{n} \eta_j^2} \qquad (17)$$

where E(·) is the expectations operator. This result shows that it is unlikely the simple linear models encountered in the literature – of which the empirical work summarised by Dechow et al. (1999), Collins et al. (1999), Morel (2003) and Gregory et al. (2005) are good exemples – can provide dependable information about the relationship between the market value of equity and its recursion value since $\hat{\beta}$ provides a biased estimate of even the first coefficient β_1, in the series expansion

[16] In this and subsequent analysis pedagogical convenience dictates that we suppress the constant term, β_0, in the polynomial expansion for P_j without any loss of generality.

for P_j.

A more detailed assessment of the biases that are likely to arise from assuming a simple linear relationship between equity value and its determining variables can be made by imposing the conventional assumption that the error term, e_j, is an independent and identically distributed normal variate with zero mean and variance, $\sigma^2(e)$. It then follows that the variance of the difference between $\hat{\beta}$ and β_1 will be:

$$\mathrm{Var}\,(\hat{\beta} - \beta_1) = \mathrm{Var}\left\{ \frac{\sum_{j=1}^{n} \eta_j e_j}{\sum_{j=1}^{n} \eta_j^2} \right\} = \frac{\sigma^2(e)}{\sum_{j=1}^{n} \eta_j^2} \tag{18}$$

One can then compute the standardised variable:

$$z = \frac{(\hat{\beta} - \beta_1) - E(\hat{\beta} - \beta_1)}{\sqrt{\mathrm{Var}(\hat{\beta} - \beta_1)}} \tag{19}$$

which will have a standard normal distribution. Substituting equations (17) and (18) into this expression shows that the standard normal score corresponding to the null hypothesis $\hat{\beta} = \beta_1$ will be:

$$z = -\frac{\sum_{j=1}^{n} [\beta_2 \eta_j^3 + \beta_3 \eta_j^4 + \_\_\_\_ + \beta_m \eta_j^{m+1}]}{\sigma(e)\sqrt{\sum_{j=1}^{n} \eta_j^2}} \tag{20}$$

Previous analysis shows that it is likely equity prices can be represented as a polynomial of order $m \approx 20$ in their determining variables. Given this, under the hypothesis $\hat{\beta} = \beta_1$ it is highly likely that z will be large in absolute terms. This in turn means there is little reason to suppose that the null hypothesis $\hat{\beta} = \beta_1$ would not be rejected at any reasonable level of significance. In other words, there is very little prospect of the regression coefficient, $\hat{\beta}$, in a linear model of the relationship between equity value and its determining variables, providing dependable information about β_1 – or any other coefficient in the equity pricing relationship for that matter.

5. Summary and conclusions

It is now some time since Burgstahler and Dichev (1997: 212) and Penman (2001: 692) observed that the theoretical basis for empirical work on the relationship between equity value and its determining variables is extremely weak. Unfortunately, their call for more refined theoretical modelling in the area has largely been ignored. Empirical work on the relationship between the market value of equity and its determining variables continues to be based on linear models that neglect the real option effects associated with a firm's ability to modify or even abandon its existing operating activities. It is well known, however, that real options induce a convex and potentially, highly nonlinear relationship between equity values and their determining variables (Burgstahler and Dichev, 1997; Ashton et al., 2003). Given this, it is all but inevitable that when real options do impact on equity values, systematic biases will arise in empirical work based on linear valuation models. Our analysis indicates that these biases will be most pronounced for loss making firms. Unfortunately, these firms typically account for around 20% of the samples employed in empirical work (Burgsthaler and Dichev, 1997: 197; Ashton et al., 2003: 428) and so they can have a significant impact on parameter estimation.

Given the now extensive empirical evidence on this convexity issue, it is again timely to renew the call for the development of more refined analytical models of the relationship between equity value and its determining variables. There are two areas in particular where the need for enhanced modelling procedures is urgent. First, relatively little is known about the impact that the real options available to firms have on the book values of assets and liabilities and the accounting policies implemented by firms. The few papers published on this topic (Gietzmann and Ostaszewski, 1999, 2004; Ashton et al., 2003; Ashton et al., 2004) have had relatively little impact on the empirical work conducted in the area – which remains largely wedded to linear valuation models that neglect the impact which real options can have on equity values. Second, even less is known about the appropriate econometric procedures to be used in empirical work in an environment where there is a non-linear relationship between equity values and their determining variables. Suffice it to say that if econometric procedures mistakenly assume the existence of a purely linear relationship between equity values and their determining variables, then it is all but inevitable there will be problems with omitted variables and scale effects in the data. Our analysis shows, however, that these problems can be mitigated by approximating the market value of equity in terms of a polynomial expression of its determining variables – although our evidence is also that the polynomial terms will have to be carried to a fairly high order if this technique is to be satisfactory.

Appendix
Polynomial approximation to the equity valuation function

The first two Laguerre polynomials are $L_0(h) = 1$ and $L_1(h) = 1–h$ and these polynomials are orthonormal under the inner product

$$\langle f;g\rangle = \int_0^\infty f(h)g(h)e^{-h}dh$$

(Carnahan, et al., 1969: 100). Given this, consider the line of best fit to the equity valuation function in terms of these first two Laguerre polynomials:

$$P(h,1,\theta B) = h + \frac{1}{2}\int_{-1}^{1} \exp(\frac{-2B\theta h}{1+z})dz \approx \alpha_0 + \alpha_1(1-h) \tag{A1}$$

where α_0 and α_1 are known as the Fourier coefficients of the equity valuation equation, $P(h,1,\theta B)$, with respect to $L_0(h)$ and $L_1(h)$, respectively. Now, standard results show (Apostol, 1969: 29–30):

$$\alpha_0 = \langle L_0(h);P(h,1,\theta B)\rangle = \langle 1;P(h,1,\theta B)\rangle = \int_0^\infty e^{-h}[h + \frac{1}{2}\int_{-1}^{1}\exp(\frac{-2\theta Bh}{1+z})dz]dh \tag{A2}$$

and:

$$\alpha_1 = \langle L_1(h);P(h,1,\theta B)\rangle = \langle 1-h;P(h,1,\theta B)\rangle = \int_0^\infty e^{-h}(1-h)[h + \frac{1}{2}\int_{-1}^{1}\exp(\frac{-2\theta Bh}{1+z})dz]dh \tag{A3}$$

Note, however, that the expression for α_0 may be decomposed into two integrals, the first of which is

$$\int_0^\infty he^{-h}dh = 1.$$

For the second component, note that all functions under the integral sign are continuous in which case we have:

$$\frac{1}{2}\int_0^\infty\int_{-1}^{1} e^{-h}\exp(\frac{-2\theta Bh}{1+z})dzdh = \frac{1}{2}\int_{-1}^{1}\int_0^\infty \exp(\frac{-(2\theta Bh + (1+z)h)}{1+z})dhdz \tag{A4}$$

where Fubini's Theorem (Apostol, 1969: 363) allows the order of integration to be reversed. One can then evaluate this double integral as follows:

$$\frac{1}{2}\int_{-1}^{1}\int_0^\infty \exp(\frac{-(2\theta Bh + (1+z)h)}{1+z})dhdz = \frac{1}{2}\int_{-1}^{1}\frac{(1+z)}{(2\theta B + (1+z))}dz = \theta B\log(\frac{\theta B}{1+\theta B}) + 1 \tag{A5}$$

It then follows:

$$\alpha_0 = \langle L_0(h);P(h,1,\theta B)\rangle = \langle 1;P(h,1,\theta B)\rangle = \tag{A6}$$

$$\int_0^\infty he^{-h}dh + \frac{1}{2}\int_{-1}^{1}\int_0^\infty \exp(\frac{-(2\theta Bh + (1+z)h)}{1+z})dhdz = \theta B\log(\frac{\theta B}{1+\theta B}) + 2$$

Similar, though more complicated calculations show that the Fourier coefficient with respect to $L_1(h)$ is:

$$\alpha_1 = \langle L_1(h);P(h,1,\theta B)\rangle = \langle 1 - h;P(h,1,\theta B)\rangle = -1 - \frac{\theta B}{1+\theta B} - \theta B\log(\frac{\theta B}{1+\theta B}) \tag{A7}$$

Given these results, it follows that the best linear approximation to the equity valuation equation will be:

$$P(h,1,\theta B) \approx \alpha_0 + \alpha_1(1 - h) = [\theta B\log(\frac{\theta B}{1+\theta B}) + 2] + [-1 - \frac{\theta B}{1+\theta B} - \theta B\log(\frac{\theta B}{1+\theta B})](1 - h)$$

or that:

$$P(h,1,\theta B) \approx \frac{1}{1+\theta B} + [1 + \frac{\theta B}{1+\theta B} + \theta B\log(\frac{\theta B}{1+\theta B})].h \tag{A8}$$

will be the best linear approximation to the valuation function P(h,1,θB) over the semi-infinite real line. Non-linear approximations to the equity valuation function can be obtained using the higher order Laguerre polynomials based on the recursion formula (Carnahan, et al., 1969: 100):

$$nL_n(h) = (2n - 1 - h)L_{n-1}(h) - (n-1)L_{n-2}(h) \tag{A9}$$

where $L_n(h)$ is the Laguerre polynomial of order $n \geq 2$. Moreover, one can use this expression to show that the Fourier coefficient, α_n, for the equity valuation function with respect to the nth degree Laguerre polynomial, $L_n(h)$, will be:

$$\alpha_n = \frac{\theta B(1+\theta B)^n - (\theta B)^n(n+\theta B)}{n(n-1)(1+\theta B)^n} \tag{A10}$$

again provided $n \geq 2$. It then follows that the Ashton et al. (2003) equity valuation formula can be expressed in terms of the following infinite series expansion:

$$P(h,1,\theta B) = \sum_{n=0}^{\infty} \alpha_n L_n(h) \tag{A11}$$

One can illustrate these latter results by letting n = 2 in the recursion formula for the Laguerre polynomials in which case one has that the Laguerre polynomial of order two will be $2L_2(h) = (3 - h)L_1(h) - L_0(h)$ or $L_2(h) = \frac{1}{2}(h^2 - 4h + 2)$. The Fourier coefficient, α_2, for the equity valuation function with respect to $L_2(h)$ will then be:

$$\alpha_2 = \langle L_2(h);P(h,1,\theta B)\rangle = \frac{\theta B(1+\theta B)^2 - (\theta B)^2(2+\theta B)}{2.(2-1)(1+\theta B)^2} = \frac{1}{2}.\frac{\theta B}{(1+\theta B)^2} \tag{A12}$$

It then follows that the best quadratic approximation to the valuation function P(h,1,Bθ) over the semi-infinite real line will be:

$$P(h,1,\theta B) \approx \sum_{n=0}^{2} \alpha_n L_n(h) = \alpha_0 + \alpha_1(1 - h) + \frac{\alpha_2}{2}(h^2 - 4h + 2)$$

or, upon collecting terms:

$$P(h,1,\theta B) \approx \frac{\frac{3}{2}\theta B + 1}{(1+\theta B)^2} + [1 + \frac{(\theta B)^2}{(1+\theta B)^2} + \theta B\log(\frac{\theta B}{1+\theta B})].h + \frac{\frac{1}{4}\theta B}{(1+\theta B)^2}.h^2 \tag{A13}$$

Moreover, this quadratic approximation to the valuation function P(h,1,θB) will be an improvement on the linear approximation summarised earlier. This follows from the fact that the squared error from approximating P(h,1,θB) as a linear sum of the first m Laguerre polynomials is given by the squared norm:[17]

[17] A simple proof of this result is given on the Wolfram Mathworld website: http://mathworld.wolfram.com/BesselsInequality.html.

$$\left\| P(h,1,\theta B) - \sum_{n=0}^{m} \alpha_n L_n(h) \right\|^2 = \left\| P(h,1,\theta B) \right\|^2 - \sum_{n=0}^{m} \alpha_n^2 \tag{A14}$$

where

$$\left\| P(h,1,\theta B) \right\|^2 = \langle P(h,1,\theta B);P(h,1,\theta B)\rangle = \int_0^\infty e^{-h}[h + \frac{1}{2}\int_{-1}^{1} \exp(\frac{-2\theta Bh}{1+z})dz]^2 dh$$

is the squared norm of the valuation function and $\alpha_n = \langle L_n(h);P(h,1,\theta B)\rangle$ is the Fourier coefficient of the equity valuation function with respect to the nth order Laguerre polynomial. Since $\alpha_n^2 \geq 0$ for all n it follows from the right-hand side of equation (A14) that the squared error,

$$\left\| P(h,1,\theta B) - \sum_{n=0}^{m} \alpha_n L_n(h) \right\|^2,$$

declines as the order of polynomial approximation is increased. Indeed, letting $m \to \infty$ in this expression for the squared error leads to Parseval's relation, namely (Apostol, 1967: 566):

$$\left\| P(h,1,\theta B) \right\|^2 = \sum_{n=0}^{\infty} \alpha_n^2 \tag{A15}$$

Now, from equation (A10) when $n \geq 2$ we have:

$$\alpha_n^2 = \frac{\theta^2 B^2}{n^2(n-1)^2} - \frac{2\theta B}{n}.\{\frac{1}{1+(\theta B)^{-1}}\}^n.\frac{(1+\frac{\theta B}{n})}{(n-1)^2} + \{\frac{1}{1+(\theta B)^{-1}}\}^{2n}.\frac{(1+\frac{\theta B}{n})^2}{(n-1)^2} \tag{A16}$$

Consider the last term on the right-hand side of this expression, namely:

$$\{\frac{1}{1+(\theta B)^{-1}}\}^{2n}.\frac{(1+\frac{\theta B}{n})^2}{(n-1)^2} \leq \frac{(1+\theta B)^2}{(n-1)^2} \tag{A17}$$

This in turn means by the Weierstrass M test that the series expansion

$$\sum_{n=2}^{\infty} \{\frac{1}{1+(\theta B)^{-1}}\}^{2n}.\frac{(1+\frac{\theta B}{n})^2}{(n-1)^2}$$

is uniformly and absolutely convergent over the semi-infinite real line (Spiegel, 1974: 228). Similar considerations show that all other terms in the series expansion for

$$\sum_{n=2}^{\infty} \alpha_n^2$$

are uniformly and absolutely convergent over the semi-infinite real line. Given this, one can define the uniformly and absolutely convergent pseudo R-squared statistic:

$$0 \leq R_m^2 = \frac{\sum_{n=0}^{m} \alpha_n^2}{\left\| P(h,1,\theta B) \right\|^2} \leq 1 \tag{A18}$$

which gives the proportion of the squared variation in the equity valuation function associated with the best fitting mth order linear combination of Laguerre polynomials. It is then possible to use

$$\alpha_0 = \theta B\log(\frac{\theta B}{1+\theta B}) + 2,\ \alpha_1 = -1 - \frac{\theta B}{1+\theta B} - \theta B\log(\frac{\theta B}{1+\theta B})$$

and for

$$n \geq 2,\ \alpha_n = \frac{\theta B(1+\theta B)^n - (\theta B)^n(n+\theta B)}{n(n-1)(1+\theta B)^n}$$

to evaluate the expression for R_m^2 and thereby spectrally decompose the equity valuation function into its various linear and non-linear components.

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